DUALITY

Every L.P.P. is associated with another L.P.P. called the dual of the problem.
The original problem is called primal while the other is called its dual.

If the optimal solution of either problem (primal or its dual) is known then the optimal solution of the other is also available.

Consider a L.P. Problem
Find $x_1, x_2, \dots x_n$ which

$$\max.\ Z_p = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$
$$\dots \quad \dots \quad \dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

and

$$x_1, x_2, \dots x_n \geq 0.$$

The dual of the above problem is obtained by

imp.

1) Transposing the co-efficient matrix.
2) Interchanging the role of constant terms and the co-efficients of the objective fun^n.
3) Reverting the inequalities.
4) minimizing the objective function instead of maximizing it.

$\therefore$ The dual problem is

find $w_1, w_2, \ldots w_m$ for which

$\min Z_D = b_1 w_1 + b_2 w_2 + \ldots + b_m w_m$.

s.t.

$a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m \geq c_1$

$a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m \geq c_2$

$\ldots$

$a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m \geq c_n$

and $w_1, w_2, w_3 \ldots w_m \geq 0$

In matrix notation the primal and dual problems can be ~~given~~ written as follows

Primal problem

find the column vector X, which

$\max. Z_p = CX$

s.t.

$AX \leq b$

and $X \geq 0$

Dual problem

find a column vector w, which

$\min Z_D = b' w$

s.t.

$A' w \geq C'$

and $w \geq 0$.

$w = [w_1, w_2, \ldots w_m]$

& A', b', c' are transpose of A, b & C respectively

Standard form of the primal

— A linear programming problem is said to be in standard primal form if

i) All the constraints involve the sign $\leq$ if it is a problem of maximization.

2) All the constraints involve the sign $\geq$ if it is a problem of minimization.

Q. Write the dual of the problem

$\min Z = 3x_1 + x_2$

s.t.

$2x_1 + 3x_2 \geq 2$

$x_1 + x_2 \geq 1$

and $x_1, x_2 \geq 0$

Solution:

The given L.P.P. is in the standard primal form.

In matrix form the given problem can be written as

$\min Z_p = (3, 1)[x_1, x_2] = CX$

s.t.

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

ie. $AX \geq b$

The dual of the given problem is

$$\max.\ Z_D = b'w = (2,1)[w_1, w_2]$$
$$= 2w_1 + w_2.$$

s.t.

$$A'w \geq c'$$

or
$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cancel{\geq} \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

or
$$2w_1 + w_2 \leq 3$$
$$3w_1 + w_2 \leq 1$$

and $w_1, w_2 \geq 0$.

i.e.

$$\left.\begin{aligned} &\max.\ Z_D = 2w_1 + w_2 \\ &\text{s.t.} \\ &2w_1 + w_2 \leq 3 \\ &3w_1 + w_2 \leq 1 \\ &\text{and } w_1, w_2 \geq 0 \end{aligned}\right\} \text{Ans}$$

Q. Write the dual of the problem

$$\min Z = 2x_2 + 5x_3$$

s.t.

$$x_1 + x_2 \geq 2$$
$$2x_1 + x_2 + 6x_3 \leq 6$$
$$x_1 - x_2 + 3x_3 = 4$$

and $x_1, x_2, x_3 \geq 0$.

Solⁿ

First, we shall write the given problem in the standard primal form, as follows

i) Since it is a minimization problem, all the constraints must involve the sign $\geq$.

$\therefore$ Multiplying second constraint by -1, we get

$$-2x_1 - x_2 - 6x_3 \geq -6$$

ii) The third constraint is an equality, so we replace it by two constraints.

$$x_1 - x_2 + 3x_3 \leq 4 \quad \text{---} \circledast$$

and $x_1 - x_2 + 3x_3 \geq 4$

Now, multiplying first one ie. $\circledast$ by -1, we get

$$-x_1 + x_2 - 3x_3 \geq -4$$

$\therefore$ the given problem in the standard primal form is as follows.

$$\min Z_p = 2x_2 + 5x_3$$

s.t.

$$x_1 + x_2 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1 - x_2 + 3x_3 \geq 4$$

and $x_1, x_2, x_3 \geq 0$.

In matrix form

$\min Z_p = (0, 2, 5)\,[x_1, x_2, x_3]$

s.t

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ -4 \\ 4 \end{bmatrix}$$

The dual of the problem is

$$\max . Z_D = b'w$$
$$= (2, -6, -4, 4)\,[w_1, w_2, w_3', w_3'']$$

s.t.

$$\begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -6 & -3 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3' \\ w_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

ie. $\max . Z_D = 2w_1 - 6w_2 - 4w_3' + 4w_3''$

s.t:

$$w_1 - 2w_2 - w_3' + w_3'' \leq 0$$
$$w_1 - w_2 + w_3' - w_3'' \leq 2$$
$$0 \cdot w_1 - 6w_2 - 3w_3' + 3w_3'' \leq 5$$

$$w_1, w_2, w_3', w_3'' \geq 0.$$

Now, writing $w_3' - w_3'' = w_3$

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the dual problem is

$$\max.\ Z_D = 2w_1 - 6w_2 - 4w_3$$

s.t.

$$w_1 - 2w_2 - w_3 \le 0$$
$$w_1 - w_2 + w_3 \le 2$$
$$-6w_2 - 3w_3 \le 5$$

and $w_1, w_2 \geq 0$, w_3 unrestricted in sign.